MATH 5C - TEST 2 sample
(Chapter 14)

Note: This is a little longer than the test

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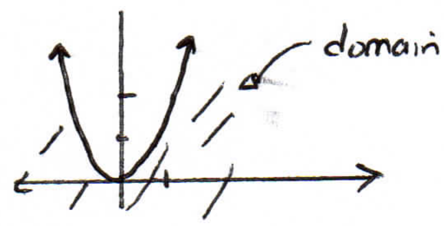
(1) Find and sketch the domain of f(x,y) = $\sqrt{2x^2 - y}$. Evaluate f(3,2). (4 points)

$2x^2 - y \geq 0$

$y \leq 2x^2$



$\boxed{f(3,2) = 4}$

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(2) Use the chain rule to find $\frac{\partial w}{\partial t}$ given w= f(x, y, z) = $x^2 \cos(3y) + \ln(z)$;

x = 7t – 4s, y=5st, z= t cos(s). (No need to simplify answer all in terms of s, t .) Show the chain rule formula you used. (5 points)

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= (2x\cos 3y)(7) + -3x^2 \sin 3y\,(5s) + \frac{1}{z}\cos s$$

$$= \boxed{14x\cos 3y - 15x^2 s \sin 3y + \frac{\cos s}{z}}$$

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(3) Find all critical points of f(x,y) =$x^2 - x^2y + 2y^2$ and classify each as yielding local max., local min., or saddle points. Show how you arrive at your conclusions. You do not need to find the functional values at the critical points. (10 points)

Critical points
$$\begin{cases} f_x = 0 & 2x - 2xy = 0 \rightarrow 2x(1-y) = 0 \Rightarrow x = 0 \text{ or } y = 1 \\ f_y = 0 & -x^2 + 4y = 0 \end{cases}$$

$x = 0$: $-x^2 + 4y = 0$, $4y = 0$, $(0,0)$

$y = 1$: $-x^2 + 4y = 0$, $-x^2 = -4$, $x = \pm 2$, $(2,1)\ (-2,1)$

$$D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2-2y & -2x \\ -2x & 4 \end{vmatrix} = (2-2y)4 - 4x^2$$
$$= 8 - 8y - 4x^2$$
$$= 4(2 - 2y - x^2)$$

Crit point	D	
(0,0)	8>0 and $f_{yy}>0$	local min
(2,1)	4(2-2-4)= -16	saddle point
(-2,1)	-16	saddle point

(4) Find the equation of the plane tangent to the surface $z = x^2e^{2y}$ at the point (3, 0, 9).

(10 points)

$z - x^2e^{2y} = 0$

If $F(x,y,z) = z - x^2e^{2y}$ then our surface is a level surface for F which means $\vec{\nabla}F$ is orthogonal to it.

$\vec{\nabla}F = \langle -2xe^{2y}, -2x^2e^{2y}, 1\rangle$

$\vec{n} = \vec{\nabla}F(3,0,9) = \langle -6, -18, 1\rangle$

plane: $\underline{-6(x-3) - 18(y-0) + 1(z-9) = 0}$ or $\underline{-6x - 18y + z = -9}$

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(5) For the function $f(x,y) = \dfrac{x^3y}{3x^6 + y^2}$, (SHOW WORK) (8 points)

(a) Find $\lim_{(x,y)\to(0,0)} f(x,y)$ along any straight line y = mx. ___0___

$$\lim_{x\to 0} \frac{x^3(mx)}{3x^6 + (mx)^2} = \lim_{x\to 0} \frac{mx^4}{3x^6 + m^2x^2} = \lim_{x\to 0} \frac{mx^2}{3x^4 + m^2} = \frac{0}{m^2} = 0$$

(b) Find $\lim_{(x,y)\to(0,0)} f(x,y)$ along the parabola $y = x^2$. ___0___

$$\lim_{x\to 0} \frac{x^3x^2}{3x^6 + (x^2)^2} = \lim_{x\to 0} \frac{x^5}{3x^6 + x^4} = \lim_{x\to 0} \frac{x}{3x^2 + 1} = 0$$

(c) Find $\lim_{(x,y)\to(0,0)} f(x,y)$ along the curve $y = x^3$. ___$\frac{1}{4}$___

$$\lim_{x\to 0} \frac{x^3x^3}{3x^6 + (x^3)^2} = \lim_{x\to 0} \frac{x^6}{3x^6 + x^6} = \lim_{x\to 0} \frac{1}{4} = \frac{1}{4}$$

(d) What can be said about $\lim_{(x,y)\to(0,0)} f(x,y)$? ___DNE___

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(6) Given $z = y^2 - 5xy$. If x changes from 3 to 3.1 and y changes from 5 to 4.8, find and compare the values of Δz and dz.

$f_x = -5y \quad f_y = 2y - 5x$

(8 points)

$\Delta z = f(3.1, 4.8) - f(3,5) = [4.8^2 - 5(3.1)(4.8)] - [25 - 75] = -1.36$

Compare

These should be close

$dz = f_x(3,5)\Delta x + f_y(3,5)\Delta y$

$= -25(0.1) + -5(-0.2)$

$= -2.5 + 1 = -1.5$

(7) The temperature at a point (x, y) on a metal plate in the xy-plane is $T(x,y) = 20-4x^2 - y^2$ degrees Celsius where x and y are measured in centimeters.

(15 points)

(a) Find the rate of change of the temperature at (2,-3) as we move towards (5, -2). Is the temperature getting warmer or colder? (Show correct units in answer.)

$T_x = -8x \qquad T_y = -2y \qquad \vec{\nabla}T(x,y) = \langle -8x, -2y \rangle \qquad \vec{\nabla}T(2,-3) = \langle -16, 6 \rangle$

direction: (2,-3) towards (5,-2) $\Rightarrow \langle 3,1 \rangle$ Need unit vector $\vec{u} = \frac{1}{\sqrt{10}}\langle 3,1 \rangle$

$D_{\vec{u}}T(2,-3) = \vec{\nabla}T(2,-3) \cdot \vec{u} = \langle -16, 6 \rangle \cdot \frac{1}{\sqrt{10}}\langle 3,1 \rangle = \boxed{\frac{-42}{\sqrt{10}} \, {}^\circ C/cm \quad \text{getting colder}}$

(b) In what direction from (2, -3) does the temperature increase most rapidly? What is the rate of increase?

The temp increases most rapidly in the direction of $\vec{\nabla}T(2,-3) \Rightarrow \langle -16, 6 \rangle$

The rate if $D_{\vec{u}}T(2,-3)$ in that direction, shortcut $\boxed{\|\vec{\nabla}T(2,-3)\| = \sqrt{292} = 2\sqrt{73} \, {}^\circ C/cm}$

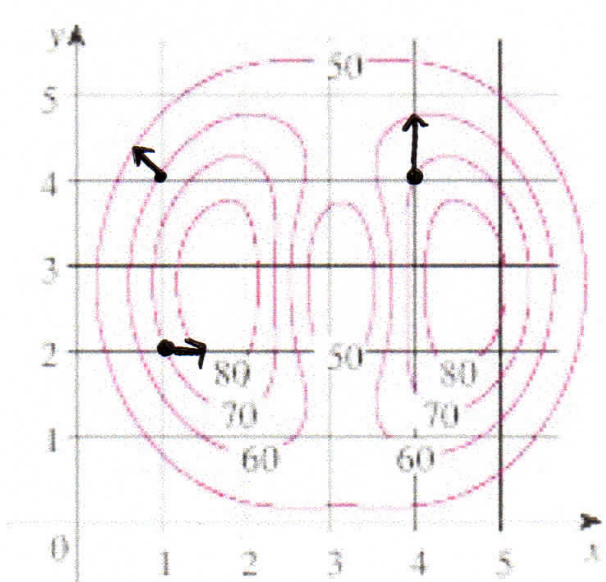
(c) Find $T_y(-1,1)$. Describe the physical meaning of this in terms of temperature on the metal plate.

$T_y(-1,1) = -2(1) = \boxed{-2 \, {}^\circ C/m}$

incorrect to say "along y axis" or "toward y axis"

This means that if moving from (-1,1) in the positive y direction the instantaneous rate of change of Temp is $-2\,{}^\circ C/m$.

(8) Given the following level curves for an unknown function f(x,y),



Estimate the following. Show work on b,c,d. (8 points)

(a) f(1,2) 70

(b) $\left.\frac{\partial f}{\partial x}\right|_{(1,2)}$ ≈ 20 $\qquad \frac{80-70}{1/2} = 20$

(c) $f_y(4,4)$ ≈ -15

$\frac{60-70}{2/3} = \frac{-10}{2/3} = -15$

(d) $D_{\vec{u}}f(1,4)$ where $\vec{u} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ -20

$\frac{50-60}{1/2} = \frac{-10}{1/2}$

(9) Match the following functions with their graphs:

(8 points)

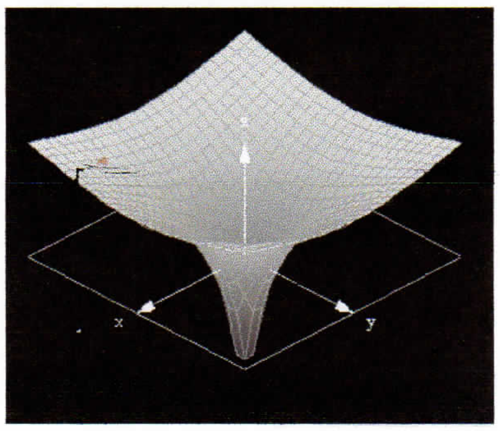
a) $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ B

(b) $f(x,y) = \cos(e^x+e^y)$ D

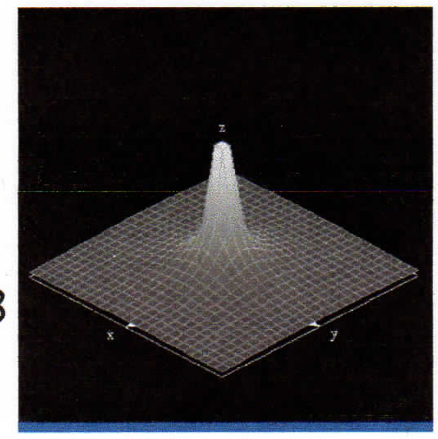
c) $f(x,y) = \ln(x^2+y^2)$ A

(d) $f(x,y) = \cos(xy)$ C

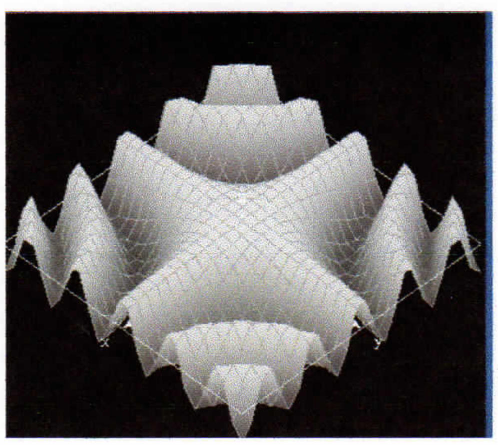
A



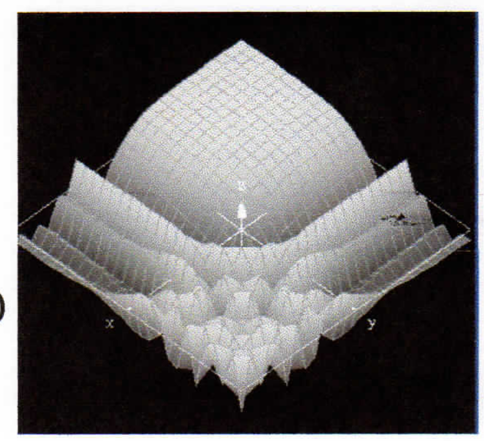
B



C



D



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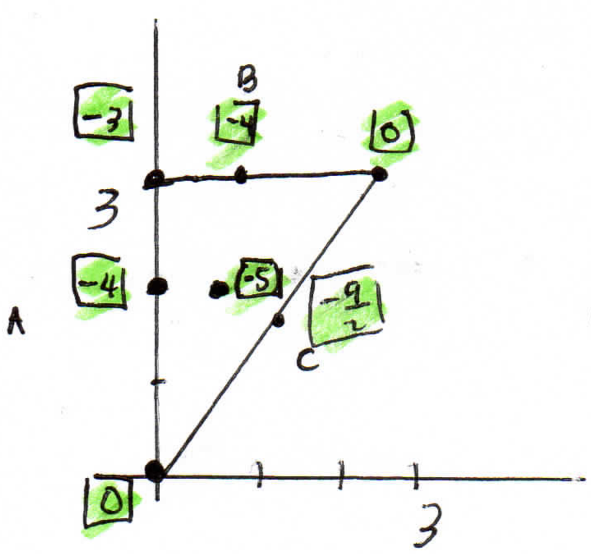
(10) Find the absolute extreme values of $f(x,y) = x^2 + y^2 - 2x - 4y$ on the region bounded by $y=x$, $y=3$, and $x=0$. Show all work. In particular, SHOW ALL POINTS WHICH YOU CONSIDERED AS POSSIBILITIES FOR YIELDING EXTREME VALUES.



f(x,y) conts. on a closed domain $\Rightarrow$ There must be an abs max and min which can occur at critical points or on boundary

① Critical Points $\begin{cases} f_x = 0 & 2x-2=0 \\ f_y = 0 & 2y-4=0 \end{cases}$ $(1,2)$ $f(1,2) = -5$

② Boundary: must consider 3 pieces

A: $x=0$, $0 \le y \le 3$ f(x,y) becomes $g(y) = y^2 - 4y$ $[0,3]$

$g'(y) = 2y - 4$ $y=2$

y	2	0	3
$g(y)$	-4	0	-3

B: $y=3$ f(x,y) becomes

$h(x) = x^2 - 2x - 3$ $[0,3]$

$h'(x) = 2x - 2$ $x=1$

x	1	0	3
$h(x)$	-4	-3	0

C: $y=x$

f(x,y) becomes $r(x) = 2x^2 - 6x$ $[0,3]$

$r'(x) = 4x - 6$ $x = 3/2$

x	$3/2$	0	3
$r(x)$	$-\frac{9}{2}$	0	0

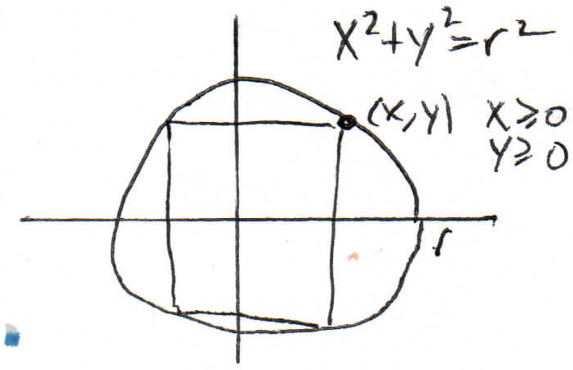
Should have compared 7 values.

MAX: 0

MIN: -5

(11) Use the method of Lagrange multipliers to show that the rectangle of largest area that can be inscribed in a circle is a square. Give the maximum rectangle area in terms of circle radius.

Show all points which you considered as possibilities for yielding extreme values.



Maximize Area Rectangle $=(2x)(2x)=4xy$

$f(x,y)=4xy$ subject to $\underbrace{x^2+y^2=r^2}_{g(x,y)}$

$\underline{\vec{\nabla} f=\lambda\vec{\nabla} g} \Rightarrow \begin{matrix} f_x=\lambda g_x \\ f_y=\lambda g_y \end{matrix} \Rightarrow \begin{cases} 4y=\lambda 2x \\ 4x=\lambda 2y \\ x^2+y^2=r^2 \end{cases} \rightarrow \begin{matrix} 4xy=\lambda 2x^2 \\ 4xy=\lambda 2y^2 \\ \lambda 2x^2=\lambda 2y^2 \end{matrix}$

$\boxed{x=\pm y}$

put $x,y\geq 0 \Rightarrow$ $\boxed{x=y}$

$\cancel{x^2+x^2=r^2}$

$\cancel{x^2=\tfrac{1}{2}r^2=y^2}$

Since $x=y$
the rectangle is a square

(12)

4. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h=f(v,t)$ are recorded in feet in the following table.

Duration (hours)

v \ t	5	10	15	20	30	40	50
10	2	2	2	2	2	2	2
15	4	4	5	5	5	5	5
20	5	7	8	8	9	9	9
30	9	13	16	17	18	19	19
40	14	21	25	28	31	33	33
50	19	29	36	40	45	48	50
60	24	37	47	54	62	67	69

Wind speed (knots)

a) $f(40,30)=31$ feet

This means that on the open sea, when winds of 40 knots have been blowing for 30 hours, the waves would be $\approx$ 31ft high

b) $\frac{\partial f}{\partial v}(40,30)\approx\frac{31-18}{40-30}=\frac{13}{10}$ or $\frac{45-31}{50-40}=\frac{14}{10}$ ft/knot

c) $f_t(40,30)\approx\frac{31-28}{30-20}=\frac{3}{10}$ or $\frac{33-31}{40-30}=\frac{2}{10}$ ft/hr.

(a) Find f(40,30) and clearly interpret the physical meaning with units.

(b) Estimate $\frac{\partial f}{\partial v}(40,30)$ and $f_t(40,30)$. Only one estimate needed for each. Interpret the physical meaning. Give proper units. Show work.

Physical meaning of partials: On the open sea, when the wind has been blowing 40 knots for 30 hours...

$\frac{\partial f}{\partial v}=1.35$ ft/knot The instantaneous rate of change of heigh relative to a change in windspeed is 1.35ft/knot. (Roughly, if the wind increases 1 knot we should expect the wave ht to increase $\approx$ 1.35ft)

$f_t(40,30)\approx 0.25$ The intantaneous rate of change of height relative to a change in the amount of time the wind has been blowing is 0.25 ft/hr.
(Roughly, if the wind continues 1 hour, the ht increases 0.25ft)

(13) Given that z implicitly represents a function of x and y in the following equation, $xz = z^2 \ln y$ find $\frac{\partial z}{\partial x}$

Implicit differentiation x, y independent.
$z = f(x,y)$

$$\frac{\partial}{\partial x} xz = \frac{\partial}{\partial x} z^2 \ln y$$

$$x\frac{\partial z}{\partial x} + z = 2z\frac{\partial z}{\partial x}\ln y \Rightarrow \frac{\partial z}{\partial x} = \frac{z}{2z\ln y - x}$$

OR

Using formula $\Rightarrow xz - z^2 \ln y = 0$

Let $F(x,y,z) = xz - z^2 \ln y$

$$\frac{\partial z}{\partial x} = \frac{-\partial F_x}{\partial F_z} = \frac{-z}{x - 2z\ln y}$$

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(14) Use differentials or a linear approximation to approximate the value of $\sqrt{26} + \sqrt[3]{63}$ without using your calculator. (You can use your calculator to check your result).

Let $f(x,y) = \sqrt{x} + \sqrt[3]{y}$

$f_x = \frac{1}{2\sqrt{x}}$

$f_y = \frac{1}{3y^{2/3}}$

easy to compute $\sqrt{25} + \sqrt[3]{64}$

$f(a,b)$

$f(25,64)$ $\quad 5+4$

9

Use this to compute $f(26,63)$

$a+\Delta x$ $\quad b+\Delta y$

$\Delta x = 1$ $\quad \Delta y = -1$

$$f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta z$$

$$f(a+\Delta x, b+\Delta y) = f(a,b) + \Delta z$$

$$f(a+\Delta x, b+\Delta y) \approx f(a,b) + dz = f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y$$

$$\sqrt{26} + \sqrt[3]{64} \approx f(25,64) + f_x(25,64)\Delta x + f_y(25,64)\Delta y$$

$$= 9 + \frac{1}{10}(1) + \frac{1}{48}(-1)$$

$$= 9.1 - \frac{1}{48} \approx 9.07917$$

Calculator value: 9.0990